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The development of the boundary layer at a rear stagnation point

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SUMMARY

This paper generalises the discussion by Proudman and Johnson of the growth of the boundary layer near a rear stagnation point by allowing the plane to have a general velocity $V(t)$, where we assume only that V is monotonic in t. It is found that their description (for constant V after an impulsive start) is valid, with only minor changes, whenever $V(t)$ is larger than, or has the same order of magnitude as t^{-1} when t is large; the forcing term due to the motion of the plane controls the flow and viscous forces are negligible except for points very close to the plane. However, when $V(t)$ is smaller than t^{-1}, viscous forces dominate throughout the fluid and are responsible for bringing the motion to the final state of rest.

1. Introduction

One of the basic flows that help in the understanding of the phenomenon of separation is that described by Proudman and Johnson [1]. In their paper, a plane that represents the rear-stagnation point of a cylinder is set in motion impulsively with a constant velocity normal to the surface of the plane. They were able to describe the development of the region of separated flow for large time t by a very simple function of a particular similarity variable. Physically, they neglected the viscous forces away from the plane so that the convection terms dominate in considering the inviscid equation in the body of the fluid. This analysis has recently been extended by Robins and Howarth [2], who found higher order terms by singular perturbation methods and showed that there is a consistent asymptotic expansion in both this outer inviscid region and also in the inner region that must exist close to the wall where the viscous forces need to be included.

In the present note we show that, if the plane is set in motion with a general velocity $V(t)$, where V is a monotonic function of the time variable t, then the description of Proudman and Johnson is valid with only minor changes whenever $V(t)$ is larger than, or has the same order of magnitude as t^{-1} when t is large. Although we assume monotonic behaviour for all time, it is sufficient to require that $V(t)$ is monotonic for t greater than some time t_0. In fact, only the similarity variable differs for each function V, the governing differential equation and solution are just those given previously in both inner and outer regions. When $V(t)$ is smaller than t^{-1}, then the viscous forces dominate and it is shown that the residual effects of the flow created when the time is finite dominate the effects determined by the forced flow as it tends to zero. The velocities decay as $\{tp(t)\}^{-1}$ for some function $p(t)$ such that $1 \ll p(t) \ll t^{\varepsilon}$ for all positive constants ε (e.g. $p(t) = \log t$), that can only be determined by numerical integration when the functional form of $V(t)$ is given for all t.

2. Basic solution

In our two-dimensional model, the fluid occupies the region $y \geqq 0$ with the x-axis lying in the plane. The velocities u and v in the x and y directions respectively are defined in terms of the stream function $\psi(x, y, t)$ by $u = \psi_y$, $v = -\psi_x$. From the beginning we assume that all physical quantities have been made non-dimensional. At time $t = 0$ the fluid is at rest, then for $t > 0$ it is set in motion such that the corresponding potential flow is given by $\psi = -xyV(t)$, with $V > 0$ for all t; in the work of both Proudman and Johnson, and Robins and Howarth, $V \equiv 1$. As with all stagnation point problems, we can write $\psi = -xF(y, t)$ to give the governing equation

$$F_{yt} - F_y^2 + FF_{yy} = \dot{V} - V^2 + F_{yyy}; \tag{1}$$

the dot represents differentiation with respect to t. The equation (1), in fact, follows directly from the Navier–Stokes equations without any boundary layer assumptions. The boundary conditions to be satisfied by F are $F(y, 0) = y$, $y \neq 0$; $F(0, t) = F_y(0, t) = 0$, $t \neq 0$; and $F_y \sim V(t)$ as $y \to \infty$; we have set $V(0) = 1$ for consistency, without any loss of generality. For small time, the behaviour has been described by Goldstein and Rosenhead [3]. We now consider solutions of equation (1) in the form

$$F(y, t) = V(t)\lambda(t)f(\eta) \text{ where } \eta = y/\lambda(t). \tag{2}$$

When (2) is substituted into the equation, there follows

$$\dot{V}f' - \frac{V\dot{\lambda}}{\lambda}\eta f'' - V^2 f'^2 + V^2 ff'' = \dot{V} - V^2 + \frac{V}{\lambda^2}f''', \tag{3}$$

where primes represent differentiation with respect to η. Away from the wall, we can follow Proudman and Johnson and ignore the final term of this equation as it represents the viscous forces. There are then three separate cases to consider depending on whether $V^2 \gg |\dot{V}|$, $V^2/|\dot{V}| = O(1)$, or $V^2 \ll |\dot{V}|$; for large t these are equivalent to the three possibilities $V \gg t^{-1}$, $V = O(t^{-1})$, or $0 < V \ll t^{-1}$. From now on we assume implicitly that t is large to avoid repetitious statements.

(i) $V \gg t^{-1}$: The first term on both the left and right hand sides of equation (3) can be neglected to show that we must have $\dot{\lambda}/\lambda$ proportional to $V(t)$. That is, there is some constant k such that $\dot{\lambda}/\lambda = kV$, which leads to

$$\lambda(t) = \exp\left\{k\int_{\alpha}^{t} V(\tau)d\tau\right\}, \tag{4}$$

where α is a positive constant that is chosen solely to ensure that the integral is well-defined. Consequently, the equation for $f(\eta)$ is

$$(f - k\eta)f'' - f'^2 = -1, \tag{5}$$

which is exactly that given by Proudman and Johnson. The boundary conditions $f(0) = 0$, $f \sim \eta$ as $\eta \to \infty$ are also unchanged in the general case, so that, when we require there to be an exponential decay of vorticity away from the plane, we must have the particular value $k = 1$ with

$$f = \eta - \frac{2}{c}(1 - e^{-c\eta}) \tag{6}$$

where c is an arbitrary constant; the improved numerical calculations of Robins and Howarth estimate c to be approximately 3.51. Therefore, the description of the flow given by Proudman and Johnson is also true in all cases where the velocity $V(t)$ is large compared with t^{-1}. A separation front divides the flow field into two regions where the normal velocity $v = F(y, t)$ is either positive and away from the plane, or negative and towards the plane. This separation front is distinguished by the non-zero value $\eta = \eta_s$ defined by $f(\eta) = 0$, and so the front moves away from the plane with velocity $\dot{\lambda}(t)$, where $\lambda(t)$ is an increasing function for all $V \gg t^{-1}$.

When, in particular, $t^{-1} \ll V(t) \ll 1$, then the motion does decay as $t \to \infty$ until eventually the fluid is completely at rest. The velocities are small as $V(t)$ when $t \to \infty$ throughout the flow field (except, as will soon be shown, for points close to the wall), which indicates that the forced flow controls the decay.

We can now turn to the inner solution, where the viscous term must be included to satisfy the no-slip condition on the wall. With $V^2 \gg |\dot{V}|$ still being satisfied, the required balance between viscous and inertial forces necessitates setting $\lambda = V^{-\frac{1}{2}}$ in (3). Therefore we introduce the inner variables by

$$F(y, t) = V^{\frac{1}{2}} g(\omega) \text{ where } \omega = V^{\frac{1}{2}} y, \tag{7}$$

to provide the differential equation

$$g''' - gg'' - 1 + g^2 = 0, \tag{8}$$

subject to the conditions $g(0) = g'(0) = 0$, $g'(\infty) = -1$. The last condition ensures that the matching can take place with the outer solution (6) where $f \sim -\eta$ as $\eta \to 0$. The solution to this equation is just the well-known Heimenz forward-stagnation-point solution, and again there is a direct generalization of the inner solution found by Robins and Howarth when $V(t) \equiv 1$. The natural variable in the inner region is ω, so that the thickness of this region is proportional to $\{V(t)\}^{-\frac{1}{2}}$ for large t; consequently, the faster the wall is moved, the thinner the region along the wall within which the viscous forces act.

When $t^{-1} \ll V(t) \ll 1$, and the fluid eventually comes to rest, then this region does grow in thickness as t increases. Therefore, although the final decay still shows that the velocity u is small as $V(t)$ when $t \to \infty$ for $\omega = O(1)$, when $y = O(1)$ we have u is small as $\{V(t)\}^{\frac{3}{2}}$; that is, the decay takes place more rapidly close to the wall.

The leading terms have now been found in both inner and outer regions, and the solution can be continued through the normal matching procedure. It is clear that the difference from that already given by Robins and Howarth is in minor detail only.

(ii) $V = t^{-1}$: Here all the terms in (3) except for that due to viscous forces, are of equal order. Within the outer region $\lambda(t)$ is still given by (4), which can be evaluated for $\eta = y/t^{k_1}$ for some constant k_1. The equation for the dependent variable, which we write as f_1 (in order to distinguish the results from those of the previous section), is

$$(f_1 - k_{1\eta})f_1'' - f_1' - f_1'^2 = -2. \tag{9}$$

A solution to this equation that satisfies $f_1(0) = 0$ and $f_1' \sim 1$ with exponential error as $\eta \to \infty$ is only possible when $k_1 = 1$; the details are similar to those in Proudman and Johnson, and lead to

$$f_1 = \eta - \frac{3}{c_1}(1 - e^{-c_1\eta}); \tag{10}$$

c_1 is some constant. It can be checked that the viscous term F_{yyy} is still small compared with the other terms, so that the main assumption for the outer region is still valid.

There is still no change from the basic description given by Proudman and Johnson. Here the separation front moves away from the plane with a constant velocity, though the velocities within the fluid are small as t^{-1} when $t \to \infty$.

Within the inner region, where now viscous, inertial and time dependent terms will all be in balance, we write

$$F = t^{-\frac{1}{2}}g_1(\zeta) \text{ where } \zeta = yt^{-\frac{1}{2}} \tag{11}$$

and ζ is the standard variable for viscous diffusion. The equation for g_1 is

$$g_1''' + \tfrac{1}{2}\zeta g_1'' + g_1' + g_1'^2 - g_1 g_1'' - 2 = 0, \tag{12}$$

with the boundary conditions $g_1(0) = g_1'(0) = 0$, and $g_1'(\infty) = -2$ to match with the outer solution (10) where $f_1 \sim -2\eta$ as $\eta \to 0$. This equation has a unique solution that displays an exponential decay in the vorticity as $\zeta \to \infty$; the velocity profiles display no real differences from those already known from the Hiemenz equation (8).

Although all three terms are in balance within the inner region here, it can easily be seen that this is not true within the total flow field. If it were so, then the expansion (11) would be valid in both regions, and it would be necessary to solve the equation (12) with the condition at infinity replaced by $g_1'(\infty) = 1$. However, there is then no solution, and so the two regions do exist.

(iii) $V \ll t^{-1}$. The equation corresponding to (3) now becomes $f' - (\dot{\lambda}V)/(\lambda\dot{V})\eta f'' = 1$, which has a solution in terms of the similarity variable when $\lambda = V^{-h}$ for some constant h. However, the solution of this equation for f, which is derived from the F_{yt} and $\dot{V}$ terms only of (1), is just $f \equiv \eta$. This is the potential flow which clearly does not satisfy the required conditions. Therefore, it must follow that the viscous forces can not be neglected away from the boundary when the forced flow decays so quickly, and the term F_{yyy} in equation (1) must be included throughout the flow region. In fact, because the non-linear convection terms will play a secondary role, diffusion alone must act to bring the motion to rest. It is noted that with $V \ll t^{-1}$, $\dot{\lambda}$ as given from (4) will tend to zero as $t \to \infty$, indicating that the region of reversed flow does not continue to grow but has finite dimensions. We must therefore consider this situation quite differently.

At this juncture it is instructive to examine the particular case when the function $V(t)$ is zero for t greater than some specific time T, which is $O(1)$. Because the F_{yt} and F_{yyy} terms will balance for large t, it is appropriate to introduce the diffusion variable $\zeta = yt^{-\frac{1}{2}}$, and when we further define the new dependent variable $H(\zeta, t)$ by $F = t^{\frac{1}{2}}H$, we have, from (1) with $V \equiv 0$, the equation

$$H_{\zeta\zeta\zeta} + \tfrac{1}{2}\zeta H_{\zeta\zeta} - tH_{\zeta t} + t(H_\zeta^2 - HH_{\zeta\zeta}) = 0. \tag{13}$$

When all the terms of (13) have equal magnitude for large t, with both viscous and convection terms in balance, we must set $H(\zeta, t) = t^{-1}h_1(\zeta)$. Therefore h_1 then satisfies

$$h_1''' + \tfrac{1}{2}\zeta h_1'' + h_1' + h_1'^2 - h_1 h_1'' = 0,$$

which is similar to (12); however, this equation has no solution that satisfies the necessary conditions $h_1(0) = h_1'(0) = 0$ plus h_1 tends exponentially to a positive constant as $\zeta \to \infty$.

However, when we write, for example,

$$H(\zeta, t) = \frac{h(\zeta)}{t(\log t)^\alpha}, \tag{14}$$

where $\zeta = O(1)$ and α is some positive constant, the non-linear terms can be neglected for large t and h satisfies the linear equation $h''' + \frac{1}{2}\zeta h'' + h' = 0$; this equation is derived from the F_{yt} and F_{yyy} terms alone in (1). The solution that satisfies all the conditions is

$$h = \kappa(1 - e^{-\zeta^2/4}) \tag{15}$$

for constant κ. In fact, we could write

$$H(\zeta, t) = \{tp(t)\}^{-1}h(\zeta), \tag{16}$$

for any function $p(t)$ such that $1 \ll p(t) \ll t^\varepsilon$ when t is large for all positive constants ε (however small), in place of (14) and still have $h(\zeta)$ given by (15). (For example, $p(t)$ could be $\log(\log t)$ and show a slower decay than that given by (14).) A representation of this type indicates the slowest decay possible in which the viscous forces alone bring the fluid to rest, and it also provides the slowest decay possible compared to the t^{-1} decay for the velocities found in (ii) when $\zeta = O(1)$, as should be expected for a continuous change as $V(t)$ decreases in magnitude for large t.

When H has a simple algebraic decay in t such that we write $H = t^{-\gamma}k(\zeta)$ for large t, then (13) shows that k must satisfy $k''' + \frac{1}{2}\zeta k'' + \gamma k' = 0$ when $\gamma > 1$. Now solutions of this equation that satisfy the no-slip conditions on $\zeta = 0$, and also show k' to be exponentially small for large ζ, only exist when γ is a positive integer; the smallest value is therefore $\gamma = 2$.

Because no information concerning the nature of the flow for finite times has yet been included, there is no assurance that one or the other of these solutions represents the actual solution of the physical problem for large times. However, a model displaying certain parallels is that of linear diffusion with arbitrary initial data. That is, we solve the equation $F_{yt} = F_{yyy}$ subject to $F = F_y = 0$ on $y = 0$, $F_y \to \infty$ as $y \to \infty$, plus $F_y = -U(y)$ at $t = 0$, where $U(y) > 0$ is the initial velocity profile. The equation is an approximate form for (1), but it does include the terms that are known to be dominant for large time, and, by incorporating an initial condition, it attempts a description for all time rather than for large times alone. The information gained by such an approximation must, of course, be treated with caution because the non-linear self-convection terms are ignoted, and they do have a fundamental role for finite time when the separation will occur. Nevertheless, the solution has been gained to the above problem and is given by

$$F_y = -\frac{1}{(\pi t)^{\frac{1}{2}}} \int_0^\infty U(\alpha) \exp\left(-\frac{\alpha^2}{4t}\right) \sinh\left(\frac{y\alpha}{2t}\right) d\alpha.$$

When $\zeta = O(1)$ and $t \gg 1$, then asymptotic methods show $F_y \propto \zeta e^{-\zeta^2/4} t^{-1}$. Hence this approximation indicates a t^{-1} decay for the velocity u for all initial profiles $U(y)$; the influence of the specific function is felt only through the multiplicative constant $\int_0^\infty \alpha U(\alpha)\, d\alpha$. Now it is already known that the t^{-1} decay is not permitted for the full non-linear problem, but, in contrast to the $\{tp(t)\}^{-1}$ decay defined above, the t^{-2} (or any faster) decay in u suggested in the previous paragraph can be reasonably rejected.

If the first term is given by (14) then higher terms can be found for the expansion of H in the normal manner. There will be an infinity of terms of the form $t^{-1}(\log t)^{-\mu} h_\mu(\zeta)$ for increasing values of $\mu > \alpha$, each showing h_μ to be exponentially small as $\zeta \to \infty$. All these terms will therefore tend to zero slower than $t^{-1-\varepsilon}$ as $t \to \infty$ for all $\varepsilon > 0$, and so provide a generality of solution that would seem to be required by the infinity of behaviours for finite times for which this represents the final decay. This statement is also true for all functions $p(t)$ as defined above.

It is reasonable to conclude that, in general, whenever the forcing term shows $V \ll t^{-1}$ as $t \to \infty$ (except possibly when V is itself of the form (16), for which a simple modification is possible) then the homogeneous part of the solution is larger than the inhomogeneous part. It is the residual effects of the flow for finite time that dominate rather than those due to the forcing term. The final decay takes place between the viscous and transient terms and shows the velocities small as $\{tp(t)\}^{-1}$ where $p(t)$ is some function such that $1 \ll p(t) \ll t^\varepsilon$ for all $\varepsilon > 0$. As long as $V(t)$ is large enough for $t = O(1)$, separation will occur and the region of reversed flow will move out into the body of the fluid. However, for large times the boundary of this region comes to rest as noted earlier, and ultimately viscous forces work to bring the velocities to zero.

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